Rigidity of Formations with Additional Subtended-angle Constraints

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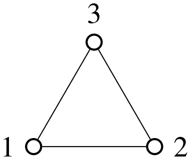
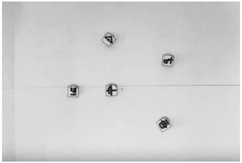
Multi-agent formation

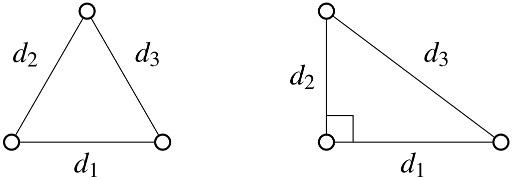
- A geometrical shape formed by multiple agents in a space
- Represented by a graph (vertices=agents)
- Examples:
  - A group of vehicles
  - A group of flying multi-copters





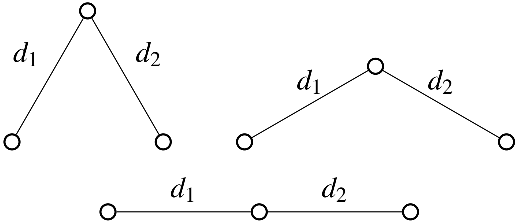
Constraints for a formation

- Distance constraints could define a formation.
- $d_1 = d_2 = d_3 = 1$: a regular triangle formation
- $d_1 = 4$, $d_2 = 3$, $d_3 = 5$: a right triangle formation



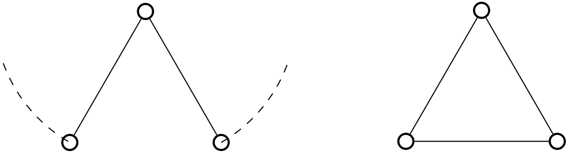
Constraints for a formation

- Insufficient constraints may not define a unique formation shape.
- $d_1 = d_2 = 1$: infinitely many non-congruent shapes satisfying the distance constraints



Condition for unique formation shape

- Rigidity is a condition so that the formation of interest is (at least locally) uniquely defined under given distance constraints.
- (Distance) rigidity has been widely studied in the literature.[1]



(a) Flexible formation (b) Rigid formation

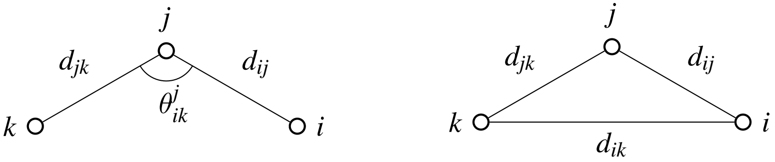
[1] L. Asimow and B. Roth. "The rigidity of graphs". In: *Transactions of the American Mathematical Society* 245.11 (1978), pp. 279–289; B. Hendrickson. "Conditions for unique graph realizations". In: *SIAM Journal on Computing* 21.1 (1992), pp. 65–84; B. D. O. Anderson et al. "Rigid graph control architectures for autonomous formations". In: *IEEE Control Systems Magazine* 28.6 (2008), pp. 48–63.

Motivation: Main idea

- By the law of cosines, we have

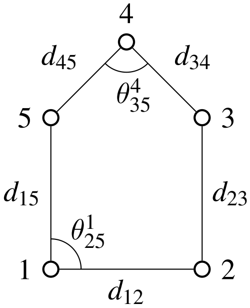
$$(d_{ik})^2 = (d_{ij})^2 + (d_{jk})^2 - 2d_{ij}d_{jk}\cos\theta^j_{ik}.$$

- d_{ij}, d_{jk}, and θ^j_{ik} determine the unique distance between agents.
- An angle constraint can be equivalently converted to a distance constraint.



Mixed constraints

- Suppose that there is an angle constraint in addition to the existing distance constraints.
- If the given constraints are satisfied, would the formation shape be unique?



Problem

- Find an exact condition using a new concept of rigidity to distinguish whether the given mixed constraints can define a unique formation shape.

Notation and terminologies

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of all vertices (agents) of the graph $\mathcal{G}$, and $\mathcal{E}$ is the set of all edges of the same graph.
- Let $\mathbf{p}_i \in \mathbb{R}^2$ denote the position vector of agent i.
- $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_{|\mathcal{V}|}) \in \mathbb{R}^{2|\mathcal{V}|}$ is called a *realization* of $\mathcal{G}$ in $\mathbb{R}^2$.
- $(\mathcal{G}, \mathbf{p})$ is called a *framework (formation)*.
- $\mathbf{p}_{ij} = \mathbf{p}_i - \mathbf{p}_j$.

Assumption

Realizability: There exists at least one realization satisfying the given distance and angle constraints.

Notation and terminologies

- For two realizations $\mathbf{p}$ and $\mathbf{q}$ of $\mathcal{G}$, we say $\mathbf{p}$ and $\mathbf{q}$ are *congruent* if $\|\mathbf{p}_{ij}\| = \|\mathbf{q}_{ij}\|$ for all $i, j \in \mathcal{V}$.
- Two frameworks $(\mathcal{G}, \mathbf{p})$ and $(\mathcal{G}, \mathbf{q})$ are said to be congruent if $\mathbf{p}$ and $\mathbf{q}$ are congruent.
- If $\|\mathbf{p}_{ij}\| = \|\mathbf{q}_{ij}\|$ for all $\{i, j\} \in \mathcal{E}$, we say $(\mathcal{G}, \mathbf{p})$ and $(\mathcal{G}, \mathbf{q})$ are *equivalent*.

Distance rigidity of a framework

Definition (Asimow and Roth, 1978)

A given framework $(\mathcal{G}, \mathbf{p})$ is *rigid* in $\mathbb{R}^2$ if there exists a neighborhood $\mathcal{B}_{\mathbf{p}} \subseteq \mathbb{R}^{2|\mathcal{V}|}$ of $\mathbf{p}$ such that each framework $(\mathcal{G}, \mathbf{q})$, $\mathbf{q} \in \mathcal{B}_{\mathbf{p}}$, equivalent to $(\mathcal{G}, \mathbf{p})$ is congruent to $(\mathcal{G}, \mathbf{p})$.

(Asimow and Roth, 1978)[2]

[2] Asimow and Roth, “The rigidity of graphs”.

Subtended-angle constraints

- For a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, consider some distance constraints $d_{ij} \in [0, +\infty)$ and subtended-angle constraints $\theta^l_{km} \in [0, \pi]$.
- d_{ij}: distance constraint assigned to $\{i, j\} \in \mathcal{E}$
- θ^l_{km}: angle constraint assigned to $\{k, l\}$ and $\{l, m\}$ in $\mathcal{E}$
- Let $\mathcal{A} = \left\{ (i, \{j, k\}) \;\middle|\; \theta^i_{jk} \text{ is assigned to } \{i, j\},\ \{i, k\} \in \mathcal{E} \right\}$.

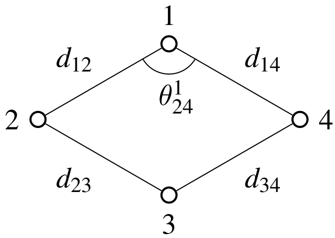
Assumption

1. *The distance constraints are assigned to all edges in $\mathcal{E}$.*
2. *The angle constraints are assigned to some neighboring edge pairs.*

Subtended-angle constraints

Example:

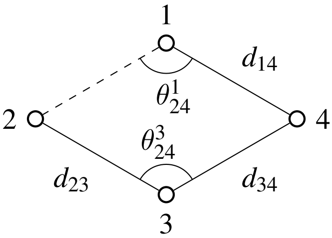
- Distance constraints: $d_{12}, d_{14}, d_{23}, d_{34}$
- Angle constraint: θ_{24}^{1}
- $\mathcal{A} = \{(1, \{2, 4\})\}$.



Subtended-angle constraints

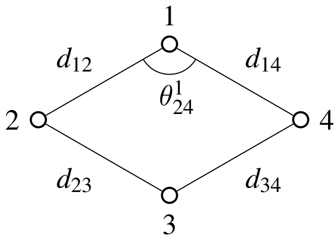
Example: (Exception)

- Distance constraints: d_{14}, d_{23}, d_{34}
- Angle constraint: θ^1_{24}, θ^3_{24}.
- $\mathcal{A} = \{(1, \{2, 4\}), (3, \{2, 4\})\}$.



- Question: Is it rigid? or not rigid?

Subtended-angle constraints



- Only consider the left cases. The right cases could be defined similarly; but a more general setup is required.

Strong equivalence

Definition (strong equivalence)

Under a given angle set $\mathcal{A}$ of a graph $\mathcal{G}$ with $|\mathcal{V}| \geq 3$, two frameworks $(\mathcal{G}, \mathbf{p})$ and $(\mathcal{G}, \mathbf{q})$ are said to be *strongly equivalent* if the following two conditions hold:

- $\forall \{v, w\} \in \mathcal{E}, \; \|\mathbf{p}_{vw}\| = \|\mathbf{q}_{vw}\|$,
- $\forall (i, \{j, k\}) \in \mathcal{A}, \; \angle \mathbf{p}^i_{jk} = \angle \mathbf{q}^i_{jk}$.

Notation:

- $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$ for two vectors $\mathbf{v}_i$ and $\mathbf{v}_j$.
- $\angle \mathbf{v}^i_{jk} \in [0, \pi]$: the angle subtended by $\mathbf{v}_{ji}$ and $\mathbf{v}_{ki}$ for three vectors $\mathbf{v}_i$, $\mathbf{v}_j$, and $\mathbf{v}_k$ provided that neither $\mathbf{v}_{ji}$ nor $\mathbf{v}_{ki}$ is empty.

Strong equivalence & Weak rigidity

Remark

If two frameworks $(\mathcal{G}, \mathbf{p})$ *and* $(\mathcal{G}, \mathbf{q})$ *are strongly equivalent, then they are equivalent, but the converse is not true.*

- If $\mathcal{A} = \emptyset$, then the definition of strong equivalence agrees to the definition of equivalence.

Definition (weak rigidity)

A given framework $(\mathcal{G}, \mathbf{p})$ with an associated angle set $\mathcal{A}$ is *weakly rigid* in $\mathbb{R}^2$ if there exists a neighborhood $\mathcal{B}_{\mathbf{p}} \subseteq \mathbb{R}^{2|\mathcal{V}|}$ of $\mathbf{p}$ such that each framework $(\mathcal{G}, \mathbf{q})$, $\mathbf{q} \in \mathcal{B}_{\mathbf{p}}$, strongly equivalent to $(\mathcal{G}, \mathbf{p})$ is congruent to $(\mathcal{G}, \mathbf{p})$.

- If $\mathcal{A} = \emptyset$, the definition of weak rigidity coincides with the definition of classical distance rigidity.

A modified graph

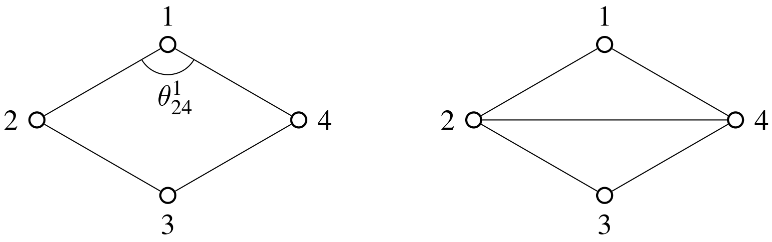
Let $\bar{\mathcal{G}}$, $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$, be a graph modified from $\mathcal{G}$, $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, in such a way that:

- $\bar{\mathcal{V}} = \mathcal{V}$,
- $\bar{\mathcal{E}} = \{\{i,j\} \mid \{i,j\} \in \mathcal{E} \vee \exists k \in \mathcal{V}$ s.t. $(k, \{i,j\}) \in \mathcal{A}\}$.
- $\bar{\mathcal{A}} = \emptyset$.

A modified graph

Example:

- $\mathcal{V} = \{1,2,3,4\}$, $\mathcal{E} = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$, $\mathcal{A} = \{(1,\{2,4\})\}$.
- $\bar{\mathcal{V}} = \{1,2,3,4\}$, $\bar{\mathcal{E}} = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\},\{2,4\}\}$, $\bar{\mathcal{A}} = \emptyset$.



(a) $\mathcal{G}$ (b) $\bar{\mathcal{G}}$

Main result

Theorem

A framework $(\mathcal{G}, \mathbf{p})$ with an angle set $\mathcal{A}$ is weakly rigid in $\mathbb{R}^2$ if and only if $(\bar{\mathcal{G}}, \mathbf{p})$ is rigid in $\mathbb{R}^2$.

Proof: (Sufficient condition) Suppose that $(\bar{\mathcal{G}}, \mathbf{p})$ is rigid in $\mathbb{R}^2$. Then, there exists a neighborhood $\mathcal{B}_{\mathbf{p}} \subseteq \mathbb{R}^{2|\bar{\mathcal{V}}|}$ of $\mathbf{p}$ such that for any $\mathbf{q} \in \mathcal{B}_{\mathbf{p}}$, if $(\bar{\mathcal{G}}, \mathbf{p})$ and $(\bar{\mathcal{G}}, \mathbf{q})$ are equivalent, then $\mathbf{p}$ and $\mathbf{q}$ are congruent. Now consider $(\mathcal{G}, \mathbf{q})$ with $\mathbf{q} \in \mathcal{B}_{\mathbf{p}}$ such that $(\mathcal{G}, \mathbf{p})$ and $(\mathcal{G}, \mathbf{q})$ are strongly equivalent. Then, it is true that

$$\forall \{v, w\} \in \mathcal{E}, \ \|\mathbf{p}_{vw}\| = \|\mathbf{q}_{vw}\|, \quad (1)$$

$$\forall (i, \{j, k\}) \in \mathcal{A}, \ \angle \mathbf{p}^i_{jk} = \angle \mathbf{q}^i_{jk}. \quad (2)$$

Then, we can state that

$$\forall (i, \{j, k\}) \in \mathcal{A}, \ \|\mathbf{p}_{jk}\| = \|\mathbf{q}_{jk}\|, \quad (3)$$

Main result

because we have

$$\begin{aligned}\|\mathbf{p}_{jk}\|^2 &= \|\mathbf{p}_{ij}\|^2 + \|\mathbf{p}_{ik}\|^2 - 2\|\mathbf{p}_{ij}\|\|\mathbf{p}_{ik}\| \cos \angle \mathbf{p}^i_{jk} \\ &= \|\mathbf{q}_{ij}\|^2 + \|\mathbf{q}_{ik}\|^2 - 2\|\mathbf{q}_{ij}\|\|\mathbf{q}_{ik}\| \cos \angle \mathbf{q}^i_{jk} \\ &= \|\mathbf{q}_{jk}\|^2,\end{aligned}$$

from (1) and (2). Therefore, we have $\|\mathbf{p}_{ij}\| = \|\mathbf{q}_{ij}\|$ for all $\{i, j\} \in \bar{\mathcal{E}}$ from (1) and (3), which means that $(\bar{\mathcal{G}}, \mathbf{p})$ and $(\bar{\mathcal{G}}, \mathbf{q})$ are equivalent, and $\mathbf{p}$ and $\mathbf{q}$ are congruent from rigidity of $(\bar{\mathcal{G}}, \mathbf{p})$. Consequently, we have shown that there exists a neighborhood $\mathcal{B}_{\mathbf{p}}$ in which $\mathbf{p}$ and $\mathbf{q}$ are congruent under strong equivalence of $(\mathcal{G}, \mathbf{p})$ and $(\mathcal{G}, \mathbf{q})$, which means that $(\mathcal{G}, \mathbf{p})$ is weakly rigid in $\mathbb{R}^2$ (by the definition of weakly rigidity).

Main result

(Necessary condition) Suppose that $(\mathcal{G}, \mathbf{p})$ is weakly rigid in $\mathbb{R}^2$. Then, there exists a neighborhood $\mathcal{B}_{\mathbf{p}} \subseteq \mathbb{R}^{2|\mathcal{V}|}$ of $\mathbf{p}$ such that for each $\mathbf{q} \in \mathcal{B}_{\mathbf{p}}$, if $(\mathcal{G}, \mathbf{p})$ and $(\mathcal{G}, \mathbf{q})$ are strongly equivalent, then $\mathbf{p}$ and $\mathbf{q}$ are congruent. Consider an arbitrary $\mathbf{q} \in \mathcal{B}_{\mathbf{p}}$ such that $(\bar{\mathcal{G}}, \mathbf{p})$ and $(\bar{\mathcal{G}}, \mathbf{q})$ are equivalent. Thus, we have

$$\forall \{i,j\} \in \bar{\mathcal{E}}, \ \|\mathbf{p}_{ij}\| = \|\mathbf{q}_{ij}\|. \tag{4}$$

Then, it is true that

$$\forall (i, \{j,k\}) \in \mathcal{A}, \ \angle \mathbf{p}^i_{jk} = \angle \mathbf{q}^i_{jk}, \tag{5}$$

because we have

$$\begin{aligned}\angle \mathbf{p}^i_{jk} &= \arccos\left[\frac{\|\mathbf{p}_{ij}\|^2 + \|\mathbf{p}_{ik}\|^2 - \|\mathbf{p}_{jk}\|^2}{2\|\mathbf{p}_{ij}\|\|\mathbf{p}_{ik}\|}\right] \\ &= \arccos\left[\frac{\|\mathbf{q}_{ij}\|^2 + \|\mathbf{q}_{ik}\|^2 - \|\mathbf{q}_{jk}\|^2}{2\|\mathbf{q}_{ij}\|\|\mathbf{q}_{ik}\|}\right]\end{aligned}$$

$$= \angle \mathbf{q}_{jk}^{i},$$

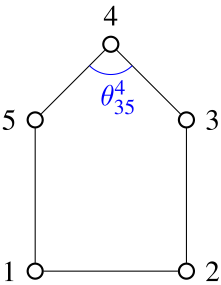
from (4). Moreover, we know that

$$\forall \{i,j\} \in \mathcal{E}, \ \|\mathbf{p}_{ij}\| = \|\mathbf{q}_{ij}\|, \tag{6}$$

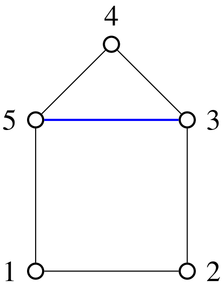
from that $\mathcal{E} \subseteq \bar{\mathcal{E}}$. Since $(\mathcal{G}, \mathbf{p})$ is weakly rigid, $\mathbf{p}$ and $\mathbf{q}$ must be congruent due to (5) and (6) under equivalence of $(\bar{\mathcal{G}}, \mathbf{p})$ and $(\bar{\mathcal{G}}, \mathbf{q})$, which means that $(\bar{\mathcal{G}}, \mathbf{p})$ is rigid in $\mathbb{R}^2$.

Example I

- $(\bar{\mathcal{G}}, \mathbf{p})$ is not rigid.
- $(\mathcal{G}, \mathbf{p})$ is not weakly rigid.



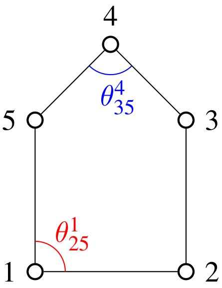
(a) $(\mathcal{G}, \mathbf{p})$



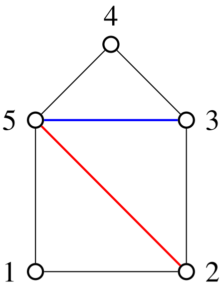
(b) $(\bar{\mathcal{G}}, \mathbf{p})$

Example II

- $(\bar{\mathcal{G}}, \mathbf{p})$ is rigid.
- $(\mathcal{G}, \mathbf{p})$ is weakly rigid.



(a) $(\mathcal{G}, \mathbf{p})$



(b) $(\bar{\mathcal{G}}, \mathbf{p})$

Example :

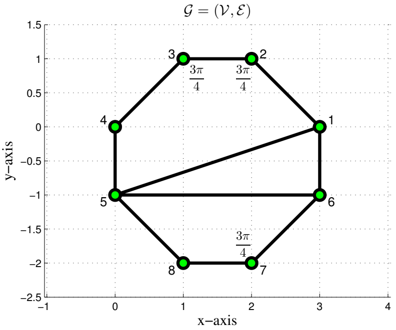


Figure. (a) An example of non-rigid but weakly rigid framework.

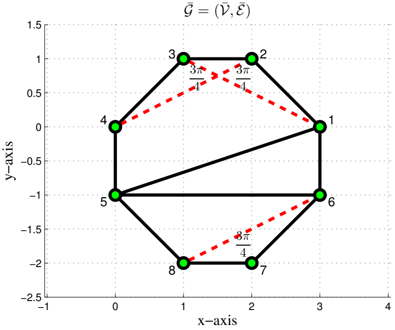


Figure. (b) An example of rigid framework.

Weak rigidity matrix (Rigidity matrix) - Credit also goes to Brian

Generic weak rigidity (Generic weak rigidity vs. weak rigidity vs. Infinitesimally weak rigidity)[3]:

- $\mathbf{p}_{ij} = \mathbf{p}_i - \mathbf{p}_j, \forall (i,j) \in \mathcal{E}$.
- $\theta_{ij}^k = (i, \{j,k\}), \forall (i, \{j,k\}) \in \mathcal{A}$; $\cos\theta_{ij}^k = \left[\frac{\|\mathbf{p}_{ik}\|^2 + \|\mathbf{p}_{jk}\|^2 - \|\mathbf{p}_{ij}\|^2}{2\|\mathbf{p}_{ik}\|\|\mathbf{p}_{jk}\|}\right]$.
- $\mathbf{e}_g \equiv \mathbf{p}_{ij}, \forall g \in \{1, ..., m_e\}$.
- $\mathbf{A}_h \equiv \cos\theta_{ij}^k, \forall h \in \{1, ..., m_a\}$.

The weak rigidity function is defined:

$\mathbf{F}_w(\mathbf{p}) \equiv [\|\mathbf{e}_1\|^2, ..., \|\mathbf{e}_{m_e}\|^2, \mathbf{A}_1, ..., \mathbf{A}_{m_a}]^T \in \mathbb{R}^{2(m_e+m_a)}$.

The weak rigidity matrix is defined as the Jacobian of the weak rigidity function: $\mathbf{R}_w(\mathbf{p}) \equiv \frac{\partial \mathbf{F}_w(\mathbf{p})}{\partial \mathbf{p}} \in \mathbb{R}^{2(m_e+m_a)\times 2|\mathcal{V}|}$.

[3] Topological concept - Given a realization, it is a weak rigid; then almost all realizations will be weak rigid. But it is not clear since we consider angles. Infinitesimally weak rigidity $\Rightarrow$ Weak rigidity

Weak rigidity matrix (Rigidity matrix) - Credit also goes to Brian

Conjecture

Let $(\mathcal{G}, \mathbf{p})$ *denote a framework in* $\mathbb{R}^2$ *with* $|\mathcal{V}| \geq 3$. *Suppose that* $\mathbf{p}$ *is generic and* $m_e \geq 1$. *Then,* $(\mathcal{G}, \mathbf{p})$ *is infinitesimally weak rigid in* $\mathbb{R}^2$ *if and only if the weak rigidity matrix* $\mathbf{R}_w(\mathbf{p})$ *has rank* $2|\mathcal{V}| - 3$ *(Issue: upto partial scaling or entire scaling ?).*

Conjecture

Let $(\mathcal{G}, \mathbf{p})$ *denote a framework in* $\mathbb{R}^2$ *with* $|\mathcal{V}| \geq 3$. *Suppose that* $\mathbf{p}$ *is generic and* $m_e = 0$. *Then,* $(\mathcal{G}, \mathbf{p})$ *is infinitesimally weak rigid in* $\mathbb{R}^2$ *if and only if the weak rigidity matrix* $\mathbf{R}_w(\mathbf{p})$ *has rank* $2|\mathcal{V}| - 4$ *(upto translations, rotations, and (partial or entire) scalings)*[a].

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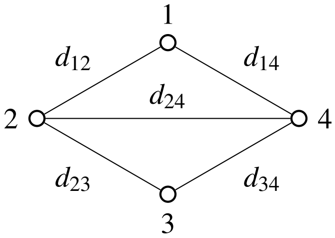
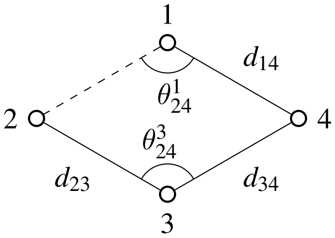
[a] T. Eren, W. Whiteley, A. S. Morse, P. N. Belhumeur, and B. D. O. Anderson. Sensor network topologies of formations with distance-direction-angle constraints. In Proc. 42nd IEEE Conf. on Decision and Control, pages 3064-3069, Maui, Hawaii USA, December 2003.

Weak rigidity matrix (Rigidity matrix) - Credit also goes to Brian

Example:
The following two examples have the same rank ($=5$).



(a) An example of rigid framework.

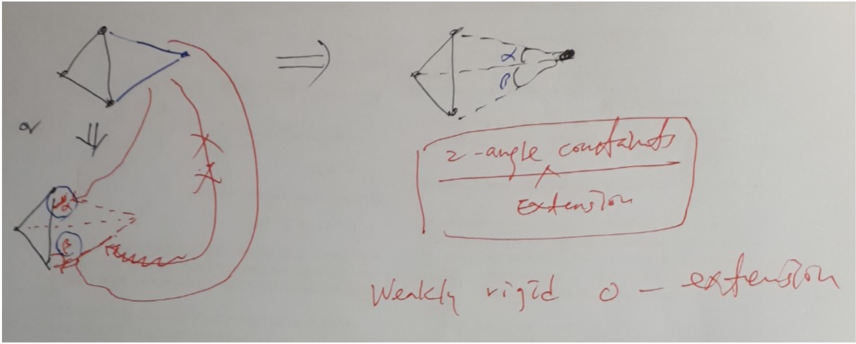


(b) An example of non-rigid but weakly rigid framework.

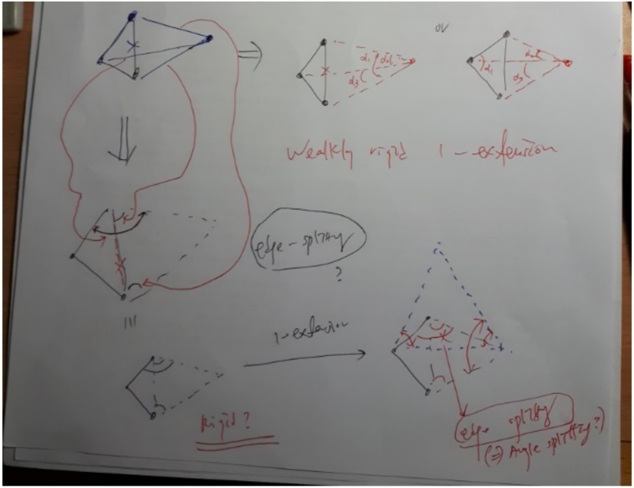
On-going Research

- How to extend the results to 3-dimensional space?
- Extension to bearing-based cases
- Extension to a large-size graph starting from K_3 by Henneberg extensions (0- and 1-extensions)
- One-to-one mapping between distance-constraints and angle-constraints

Henneberg extensions: 0-extension



Henneberg extensions: 1-extension



Extension to the 3-Dimensional Space

Conjecture

A framework $(\mathcal{G}, \mathbf{p})$ with an angle set $\mathcal{A}$ is weakly rigid in $\mathbb{R}^3$ if and only if $(\bar{\mathcal{G}}, \mathbf{p})$ is rigid in $\mathbb{R}^3$.

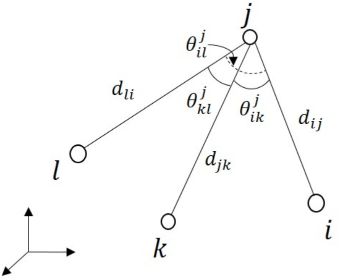


Figure. (a) An example of non-rigid but weakly rigid framework in the 3-dimensional space.

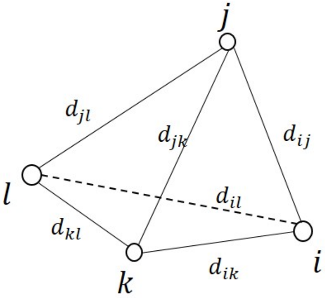


Figure. (b) An example of rigid framework in the 3-dimensional space.

Rigidity of Bearing-based Formations with Additional Subtended-angle Constraints

The following two conditions can be equivalent w.r.t. a triangular formation :

- Two bearings with another subtended angle.
- Three inter-neighbor bearings.

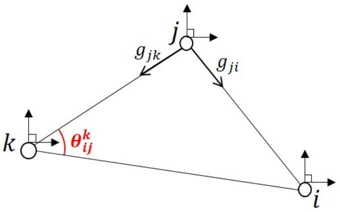


Figure. (a) An example of non-rigid but bearing-based weakly rigid framework.

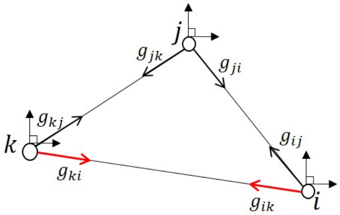


Figure. (b) An example of bearing-based rigid framework.

Conclusion

- Use a triangle as the primial component of the analysis
- One-to-one mapping between distance-constraints and angle-constraints

Main reference:
Myoung-Chul Park, Hong-Kyong Kim, Hyo-Sung Ahn, "Rigidity of Distance-based Formations with Additional Subtended-angle Constraints," *Proc. of the 17th International Conference on Control, Automation and Systems*, Je-ju, Korea, October 18-21, 2017

- On-going research:
  - Generalization in 3-dimensional space.
  - Henneberg extensions
  - Extension to the bearing-based weakly rigidity.
  - Formulation of the cases with "subtended angles without full adjacent edges".
  - Network localization

References I

[1] L. Asimow and B. Roth. “The rigidity of graphs”. In: *Transactions of the American Mathematical Society* 245.11 (1978), pp. 279–289.

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Thank you for your attention.

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