Differential LMI in Optimal Sampled-data Control

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Main notation

- Real ($\mathbb{R}$), nonnegative real ($\mathbb{R}_+$) and natural numbers ($\mathbb{N}$)
- The **trace** function is $\mathbf{tr}(\cdot)$
- For $\psi(t)$ defined for all $t \geq 0$ the value $\psi(\tau^-)$ is the limit of $\psi(t)$ as t goes to $\tau \geq 0$ from the left
- Set of bounded signals:
  - $\mathcal{L}_2$ equipped with $\|w\|_2^2 = \int_0^\infty \|w(t)\|_2^2 dt < \infty$
  - ℓ_2 equipped with $\|w\|_2^2 = \sum_{k\in\mathbb{N}} \|w[k]\|_2^2 < \infty$

Main motivation

- **Stability:** Matrix $A \in \mathbb{R}^{n \times n}$ is
  - Hurwitz (asymptotically) stable if
  $$\mathrm{Re}\{\lambda_i(A)\} < 0, \ \forall i = 1, \cdots, n$$
  - Schur (asymptotically) stable if
  $$|\lambda_i(A)| < 1, \ \forall i = 1, \cdots, n$$

- **Simple and useful relationship:** For any $h > 0$

$$A \text{ Hurwitz stable} \iff e^{Ah} \text{ Schur stable}$$

Main motivation

- Consider $h > 0$ and a polytopic convex set $\mathcal{A}_c \subset \mathbb{R}^{n \times n}$

$$\underbrace{\mathcal{A}_d}_{\textit{generic!}} = \left\{ e^{Ah} \ : \ A \in \underbrace{\mathcal{A}_c}_{\textit{polytopic}} \right\}$$

- **Robust stability (analysis):**

$$\underbrace{\forall A \in \mathcal{A}_c}_{\textit{Hurwitz}} \iff \underbrace{\forall e^{Ah} \in \mathcal{A}_d}_{\textit{Schur}}$$

Main motivation

- **Robust stability (synthesis):** Sampled-data control
  - Find a state feedback gain matrix L

$$\underbrace{e^{Ah} + \left(\int_0^h e^{A\tau} B d\tau \right) L}_{\color{red}{Schur}} \Longleftarrow \forall A \in \mathcal{A}_c$$

  - Nonlinear and nonconvex parameter dependence.
  - Very hard to solve.
  - Robust performance similar difficulty!

Main motivation

- The celebrated **Lyapunov Lemma** states that:
  - $A \in \mathbb{R}^{n \times n}$ is Hurwitz stable if and only if

$$A'P + PA < 0, \;\; P > 0$$

  - $A \in \mathbb{R}^{n \times n}$ is Schur stable if and only if

$$A'SA - S < 0, \;\; S > 0$$

- **Simple and important fact:** For any $h > 0$

$$\exists P > 0 \; s.t. \; A'P + PA < 0 \implies e^{A'h} P e^{Ah} - P < 0$$

Main motivation

- **Quadratic stability:**
  - Continuous-time polytopic system

$$A_i'P + PA_i < 0,\ \ P > 0,\ \ \forall i \in \mathcal{V}_c$$

  - Sampled-data polytopic system

$$\begin{cases} \dot{P}(t) + A_i'P(t) + P(t)A_i < 0,\ \ \forall i \in \mathcal{V}_c, t \in [0, h) \\ P(0) = P(h) = S > 0 \end{cases}$$

$$\Updownarrow$$

$$e^{A'h}Se^{Ah} - S < 0,\ \ S > 0,\ \ \forall A \in \mathcal{A}_c$$

Main motivation

- **Differential Linear Matrix Inequality (DLMI):** General form arising in $\mathcal{H}_\infty$ control design

$$\begin{bmatrix} \dot{P} + F'P + PF & PJ & G' \\ \bullet & -\gamma^2 I & 0 \\ \bullet & \bullet & -I \end{bmatrix} < 0$$

$$P(0),\ P(h) \Longleftarrow LMI$$

Importance of Two-Point Boundary Value Problem (TPBVP)

- Specific solutions $P(t)$ convert DLMI into LMI.
- Linearity with respect to the parameters.
- C. Sherer (1995) and C. Briat (2013), among others!

Bellman's principle of optimality

- The celebrated **Bellman's principle of optimality**
  - HJB equation (continuous-time) and dynamic programming (discrete-time) are the most important theoretical devices for dynamic systems optimization
  - Hard to solve due to generality!

Consider the general problem of cost evaluation

$$\min_{u \in U} \int_0^\infty f(x(t), u(t))dt$$

subject to

$$\dot{x}(t) = F(x(t), u(t)), \; x(0) = x_0$$

Bellman's principle of optimality

- Consider the sequence $\{t_k\}_{k\in\mathbb{N}}$. Bellman's principle of optimality states that

$$V(x(t_k), t_k) = \min_{u\in U}\left\{\int_{t_k}^{t_{k+1}} f(x,u)dt + V(x(t_{k+1}), t_{k+1})\right\}$$

where:

  - The function $V(x,t) : R^n \times \mathbb{R}_+ \to R$ is called cost-to-go!
  - Assuming $(x^*, u^* \in U)$ is optimal, adding terms

$$\begin{aligned}\min_{u\in U}\int_0^{\infty} f(x(t),u(t))dt &= \sum_{k\in\mathbb{N}}\int_{t_k}^{t_{k+1}} f(x(t)^*, u(t)^*)dt \\ &= V(x_0, 0)\end{aligned}$$

Bellman's principle of optimality

- Upper bounds to the optimal cost are easily generated!
- If $f(x,u) > 0, \ \forall (x,u) \neq 0$ and $f(0,0) = 0$ then
  - $V(x,t)$ is positive definite
  - $V(x(t_k), t_k) > V(x(t_{k+1}), t_{k+1})$ for all $k \in \mathbb{N}$
  - The discrete-time sequence $\{x(t_k)\}_{k \in \mathbb{N}}$ converges!
- The cost-to-go function satisfies the HJB inequality

$$\underbrace{\frac{\partial V}{\partial t} + \frac{\partial V'}{\partial x} F(x,u)}_{\frac{dV}{dt}} \leq -f(x,u), \ \ u \in U$$

  for $t \in [t_k, t_{k+1}), \ \forall k \in \mathbb{N}$.
- For time invariant systems and $t_{k+1} - t_k = h, \ \forall k \in \mathbb{N}$ a stationary solution (if any) is determined by imposing adequate boundary conditions on the time interval $t \in [0, h)$.

Bellman's principle of optimality

- Existence of a stationary (sub)optimal solution

$$v(x) \geq \min_{u \in U} \left\{ \int_0^h f(x,u)dt + v(x(h)) \right\}, \; x(0) = x \in \mathbb{R}^{n_x}$$

being optimal whenever equality holds.

  - The (sub)optimal cost is $V(x_0, 0) = v(x_0)$.
  - The (sub)optimal control is of the form $u = u(x)$.
  - A (sub)optimal solution can be determined by enforcing a particular function $v(x)$.

Sampled-data control

- Consider the LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + E_c w_c(t), \ x(0) = x_0 \\ y[k] &= C_d x(t_k) + E_d w_d[k-1] \\ z(t) &= C_c x(t) + D_c u(t) \end{aligned}$$

  - $x(\cdot) \in \mathbb{R}^{n_x}$ is the state
  - $w_c(\cdot) \in \mathbb{R}^{r_c}$ is the exogenous continuous-time input
  - $w_d[\cdot] \in \mathbb{R}^{r_d}$ is the exogenous discrete-time input ($w_d[-1]$!)
  - $y[\cdot] \in \mathbb{R}^{n_y}$ is the measured output
  - $z(\cdot) \in \mathbb{R}^{n_z}$ is the controlled output
  - $u(\cdot) \in \mathbb{R}^{n_u}$ is the control input such that

$$u \in U \iff u(t) = u(t_k), \ t \in [t_k, t_{k+1}), \ \forall k \in \mathbb{N}$$

Sampled-data control

- Dynamic output feedback sampled-data controller

$$\hat{x}[k] = \hat{A}\hat{x}[k-1] + \hat{B}y[k]$$
$$u[k] = \hat{C}\hat{x}[k-1] + \hat{D}y[k]$$

  - Initial condition $\hat{x}[-1] = 0$
  - Full order controller $\hat{x}[\cdot] \in \mathbb{R}^{n_c}$ with $n_c = n_u + n_x$!
  - Controller transfer function

$$\mathcal{C}^*(\zeta) = \hat{C}(\zeta I_{n_c} - \hat{A})^{-1}\hat{B} + \hat{D}$$

Sampled-data control

- **Sampled-data optimal control in a general framework**

$$\Downarrow$$

Hybrid Systems

$$+$$

Sampled-data control

- **Sampled-data optimal control in a general framework**

$$\Downarrow$$

Hybrid Systems

$$+$$

Bellman's Principle of Optimality

Hybrid systems

- Defining $\psi' = [x' \; u' \; \hat{x}'] \in \mathbb{R}^{n_\psi}$ with $n_\psi = n_x + n_u + n_c$, the closed-loop sampled-data system can be rewritten as, Ichicawa & Katayama (2001)

$$\begin{aligned} \dot{\psi}(t) &= F\psi(t) + J_c w_c(t) \\ z(t) &= G\psi(t) \\ \psi(t_k) &= H\psi(t_k^-) + J_d w_d[k-1] \end{aligned}$$

valid in the time interval $t \in [t_k, t_{k+1})$ for all $k \in \mathbb{N}$.

- Necessary and sufficient condition for asymptotic stability

$$\psi(t_k) \to \psi(t_{k+1}^-) \to \psi(t_{k+1}), \; \forall k \in \mathbb{N}$$

- Performance index calculation

Hybrid systems

- State space matrices

$$\underbrace{F = \begin{bmatrix} A & B & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; G = \begin{bmatrix} C_c & D_c & 0 \end{bmatrix}, \; J_c = \begin{bmatrix} E_c \\ 0 \\ 0 \end{bmatrix}}_{data}$$

and

$$\underbrace{H = \begin{bmatrix} I_{n_x} & 0 & 0 \\ \hat{D} C_d & 0 & \hat{C} \\ \hat{B} C_d & 0 & \hat{A} \end{bmatrix}, \; J_d = \begin{bmatrix} 0 \\ \hat{D} E_d \\ \hat{B} E_d \end{bmatrix}}_{controller}$$

- The process starts at $t_0 = 0$ with

$$\psi(0) = H \underbrace{\psi(0^-)}_{\psi_0} + J_d w_d[-1]$$

Hybrid systems general performance

- Given $\gamma \in \mathbb{R}_+$, consider the function

$$\rho_\gamma(\psi(0)) = \sup_{w_c \in \mathcal{L}_2,\ w_d \in \ell_2} \|z\|_2^2 - \gamma^2 \left(\|w_c\|_2^2 + \|w_d\|_2^2 \right)$$

  - **Optimal $\mathcal{H}_\infty$ sampled-data control:** For zero initial condition

$$\inf \left\{ \gamma^2 \ : \ \rho_\gamma(0) \leq 0 \right\}$$

  - **Optimal $\mathcal{H}_2$ sampled-data control:** For $r_c + r_d$ initial conditions

$$\inf \sum_{\ell=1}^{r_c+r_d} \rho_\infty(\psi_\ell(0))$$

Hybrid systems general performance

- **Theorem 1:** Let $h \in \mathbb{R}_+$, $\gamma \in \mathbb{R}_+$ and $\psi(0) \in \mathbb{R}^{n_\psi}$ be given. The hybrid linear system is asymptotically stable and

$$\rho_\gamma(\psi(0)) \leq \psi(0)' S \psi(0)$$

holds if and only if there exists $S > 0$ such that $\gamma^2 I_{r_d} > J_d' S J_d$ satisfying the two-point boundary value problem

$$\dot{P} + F'P + PF + \gamma^{-2} P J_c J_c' P + G'G \leq 0$$

subject to the initial and final conditions

$$S \geq P(0) \ , \ \ P(h) \geq \underbrace{H' \left(S^{-1} - \gamma^{-2} J_d J_d' \right)^{-1} H}_{\textit{controller}}$$

$\mathcal{H}_2$ performance

- **Corollary 1:** Let $h \in \mathbb{R}_+$ be given. The hybrid linear system is asymptotically stable and the $\mathcal{H}_2$ performance index equals the optimal solution to the convex programming problem

$$J_2 = \inf_{P(\cdot)} \left\{ \mathbf{tr}(J_c' P(h) J_c) + \mathbf{tr}(\underbrace{J_d' P(0) J_d}_{controller}) \right\}$$

subject to the differential linear matrix inequality

$$\dot{P} + F'P + PF + G'G \leq 0$$

satisfying the boundary condition

$$\underbrace{\begin{bmatrix} P(h) & H' \\ \bullet & P(0)^{-1} \end{bmatrix}}_{controller} > 0$$

$\mathcal{H}_2$ performance

- Matrix $S = P(0) > 0$ satisfies the strict Lyapunov inequality:

$$e^{F'h}H'SHe^{Fh} < S - \underbrace{\int_0^h e^{F't}G'Ge^{Ft}dt}_{\geq 0}$$

which admits a solution if and only if He^{Fh} is Schur stable.

- The optimal sampled-data $\mathcal{H}_2$ control follows from

$$\inf_{\textit{controller}, P(\cdot)} \left\{ \mathbf{tr}(J_c' P(h) J_c) + \mathbf{tr}(\underbrace{J_d' P(0) J_d}_{\textit{controller}}) \right\} \quad \rightarrow \quad \textit{CONVEX}$$

Notice that H and J_d depend on the controller matrices

$\mathcal{H}_\infty$ performance

- **Corollary 2:** Let $h \in \mathbb{R}_+$ be given. The hybrid linear system is asymptotically stable and the $\mathcal{H}_\infty$ performance index equals the optimal solution to the convex programming problem

$$J_\infty = \inf_{P(\cdot),\gamma} \gamma^2$$

subject to the differential matrix inequality

$$\dot{P} + F'P + PF + \gamma^{-2} P J_c J_c' P + G'G \leq 0$$

satisfying the boundary condition

$$\underbrace{\begin{bmatrix} P(h) & H' & 0 \\ \bullet & P(0)^{-1} & J_d \\ \bullet & \bullet & \gamma^2 I_{r_d} \end{bmatrix}}_{\textit{controller}} > 0$$

$\mathcal{H}_\infty$ performance

- A solution exists provided that $\gamma > 0$ is large enough
- The optimal sampled-data $\mathcal{H}_\infty$ control follows from

$$J_\infty = \inf_{\text{controller}, P(\cdot), \gamma} \left\{ \gamma^2 \ : \ \underbrace{\begin{bmatrix} P(h) & H' & 0 \\ \bullet & P(0)^{-1} & J_d \\ \bullet & \bullet & \gamma^2 I_{r_d} \end{bmatrix}}_{\text{controller}} > 0 \right\}$$

$\Downarrow$

CONVEX

Linearization

- Four square blocks partitioning, for all $t \in [0, h)$

$$P(t) = \begin{bmatrix} X(t) & V(t) \\ \bullet & \hat{X}(t) \end{bmatrix}, \ P(t)^{-1} = \begin{bmatrix} Y(t) & U(t) \\ \bullet & \hat{Y}(t) \end{bmatrix}$$

leads to

  - Differential LMIs
  - Boundary constraints expressed by LMIs
  - One-to-one change of variables yields the optimal controller
  - Full order controller $n_c = n_x$ due to pole / zero cancelations!

Numerical issue

- Consider the DLMI

$$\dot{P} + \mathcal{L}(P) < 0, \;\; t \in [0, h)$$

Interval $[0, h]$ with n_h segments of length $\eta = h/n_h$. Denoting $t_p = p\eta$, the piecewise linear function

$$P(t) = P_p + (P_{p+1} - P_p)\left(\frac{t - t_p}{\eta}\right), \;\; t \in [t_p, t_{p+1}]$$

with $P_p > 0$ is feasible if and only if

$$\begin{aligned} \frac{P_{p+1} - P_p}{\eta} + \mathcal{L}(P_p) &< 0 \\ \frac{P_{p+1} - P_p}{\eta} + \mathcal{L}(P_{p+1}) &< 0 \end{aligned}$$

for all $p = 0, \cdots, n_h - 1$.

Example

- Example borrowed from **Ichicawa & Katayama (2001)**.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = E_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_d' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; D_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E_d = 1$$

with $h = 1.0$. The optimal $\mathcal{H}_\infty$ sampled-data 2^{nd} order controller

$$\mathcal{C}^*(\zeta) = \frac{-0.3237\zeta^3 + 0.8679\zeta^2 - 7.188\ 10^{-6}\zeta}{\zeta^3 + 0.273\zeta^2 + 0.02648\zeta - 5.224\ 10^{-8}}$$

imposes the minimum cost $\gamma_* = 2.15$.

Example

- For $h = 1e - 4$ the delta-operator applied to the optimal $\mathcal{H}_2$ sampled-data 2^{nd} order controller

$$\mathcal{C}^*(\zeta) = \frac{7.52\,10^{-5}\zeta^3 + 7.508\,10^{-5}\zeta^2 + 1.267\,10^{-7}\zeta - 6.967\,10^{-16}}{\zeta^3 - 2\zeta^2 + 0.9995\zeta + 0.0001685}$$

provides

$$\mathcal{C}^\#(s) = \frac{-7.461\,10^{-5}s^2 - 0.746s + 0.639}{s^2 + 2.07s + 2.71}$$

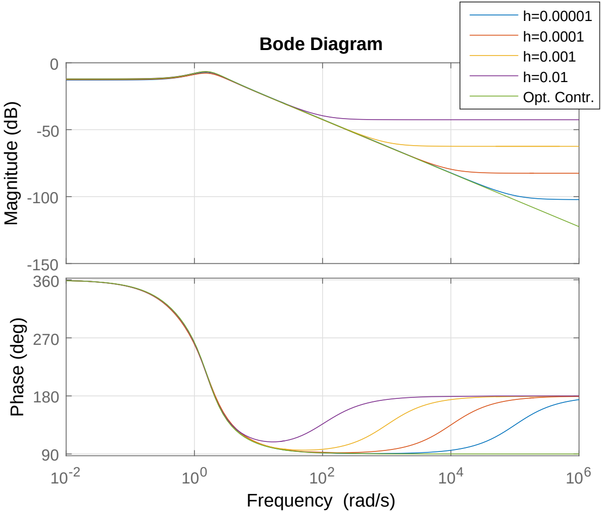
which is a good approximation of the optimal analog controller

$$\mathcal{C}(s) = \frac{-0.754s + 0.657}{s^2 + 1.82s + 2.66}$$

at low frequency.

Example

- Magnitude and phase diagrams of $\mathcal{C}^{\#}(s)$ and $\mathcal{C}(s)$



Conclusion

- **Open problems:**
  - **Dynamic output feedback control:** in the context of Markov Jump Linear Systems.
  - **Nonuniform sampling:** Optimal decision rule for the determination of $h_k \in \mathcal{T}, \ \forall k \in \mathbb{N}$, preserving stability and improving performance.

Conclusion

Thank you for your attention!!!